

# ERROR BOUNDS FOR QUASI-MONTE CARLO INTEGRATION FOR $\mathcal{L}_\infty$ WITH UNIFORM POINT SETS

SU HU AND YAN LI

**ABSTRACT.** Niederreiter [1] established new bounds for quasi-Monte Carlo integration for nodes sets with a special kind of uniformity property. Let  $(X, \mathcal{A}, \mu)$  be an arbitrary probability space, i.e.,  $X$  is an arbitrary nonempty set,  $\mathcal{A}$  a  $\sigma$ -algebra of subsets of  $X$ , and  $\mu$  a probability measure defined on  $\mathcal{A}$ . The functions considered in [1] are bounded  $\mu$ -integrable functions on  $X$ . In this note, we extend some of his results for bounded  $\mu$ -integrable functions to essentially bounded  $\mathcal{A}$ -measurable functions. So Niederreiter's bounds can be used in a more general setting.

## 1. INTRODUCTION

Let  $(X, \mathcal{A}, \mu)$  be an arbitrary probability space, i.e.,  $X$  is an arbitrary nonempty set,  $\mathcal{A}$  a  $\sigma$ -algebra of subsets of  $X$ , and  $\mu$  a probability measure defined on  $\mathcal{A}$ . Niederreiter [1] established new bounds for quasi-Monte Carlo integration for nodes sets with a special kind of uniformity property. The functions considered in [1] are bounded  $\mu$ -integrable functions on  $X$ . In this note, we extend some of his results for bounded  $\mu$ -integrable functions to essentially bounded  $\mathcal{A}$ -measurable functions.

## 2. MAIN RESULTS

Let  $\mathcal{L}_\infty(X, \mathcal{A}, \mu)$  be the set of all essentially bounded  $\mathcal{A}$ -measurable functions on  $X$ , two functions being identified if they differ only on a  $\mu$ -null set. For any  $\mathcal{A}$ -measurable function  $g$  on  $X$ ,  $\|g\|_\infty(\text{esssup}_{x \in X} |g|)$  denotes the essential supremum of  $|g|$  (see P.346 of [2]). For an extended real-valued function  $f$ , we define  $f^+ = \max\{f, 0\}$  and  $f^- = -\min\{f, 0\}$ . Notice that  $f^+ \geq 0$ ,  $f^- \geq 0$ , and  $f = f^+ - f^-$  (see P.164 of [2]). For a given nonempty subset  $\mathcal{M}$  of  $\mathcal{A}$ , let  $L_\mathcal{M}$  be linear subspace of  $\mathcal{L}_\infty(X, \mathcal{A}, \mu)$  spanned by the constant function 1 and all characteristic functions  $\chi_M$ ,  $M \in \mathcal{M}$ . For any  $f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)$ , let  $D(f, L_\mathcal{M})$  be the  $\mathcal{L}_\infty$  distance from  $f$  to  $L_\mathcal{M}$ , that is ,

$$D(f, L_\mathcal{M}) = \inf_{l \in L_\mathcal{M}} \|f - l\|_\infty.$$

The following definition can be found in P.285 of [1].

---

2000 *Mathematics Subject Classification.* Primary 11K45; Secondary 65D30.

*Key words and phrases.* Numerical integration, Quasi-Monte Carlo method, Uniform point set, Essentially bounded measurable function.

**Definition 2.1.** Let  $(X, \mathcal{A}, \mu)$  be an arbitrary probability space, let  $\mathcal{M}$  be a nonempty subset of  $\mathcal{A}$ . A point set  $\mathcal{P}$  of  $N$  elements of  $X$  is called  $(\mathcal{M}, \mu)$ -uniform if

$$\sum_{i=1}^N \chi_M(X_n) = A(M; \mathcal{P}) = \mu(M)N, \text{ for all } M \in \mathcal{M}.$$

Let  $(\underbrace{X \times \dots \times X}_N, \underbrace{\mathcal{A} \times \dots \times \mathcal{A}}_N, \underbrace{\mu \times \dots \times \mu}_N)$  be the product measurable space (see P.379 of [2]). We can view a point set  $\mathcal{P} = \{X_1, \dots, X_N\}$  as a point in  $\underbrace{X \times \dots \times X}_N$  and  $\frac{1}{N} \sum_{n=1}^N f(X_n)$  as a  $N$ -variable function on  $\underbrace{X \times \dots \times X}_N$ . Since  $f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)$ , we have  $\frac{1}{N} \sum_{n=1}^N f(X_n) \in \mathcal{L}_\infty(\underbrace{X \times \dots \times X}_N, \underbrace{\mathcal{A} \times \dots \times \mathcal{A}}_N, \underbrace{\mu \times \dots \times \mu}_N)$ .

Let

$$\mathcal{C} = \{(X_1, \dots, X_N) \in \underbrace{X \times \dots \times X}_N \mid \mathcal{P} = \{X_1, \dots, X_N\} \text{ is an } (\mathcal{M}, \mu)\text{-uniform point set}\}.$$

Since  $f(X_1, X_2, \dots, X_N) = \sum_{i=1}^N \chi_M(X_n)$  is a measurable function on  $X \times \dots \times X$ , from Definition 2.1 and Lemma 11.9 of [2], if  $\mathcal{M}$  is a countable nonempty subset of  $\mathcal{A}$ , then  $\mathcal{C}$  is a measurable set.

**Theorem 2.2.** Let  $(X, \mathcal{A}, \mu)$  be an arbitrary probability space. Let  $\mathcal{M}$  be a countable nonempty subset of  $\mathcal{A}$ . Let

$$\mathcal{C} = \{(X_1, \dots, X_N) \in \underbrace{X \times \dots \times X}_N \mid \mathcal{P} = \{X_1, \dots, X_N\} \text{ is an } (\mathcal{M}, \mu)\text{-uniform point set}\}.$$

Then for any  $f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)$ , we have

$$\text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N f(X_n) - \int_X f d\mu \right| \leq 2D(f, L_{\mathcal{M}}).$$

*Proof.* We extend Niederreiter's proof for Theorem 1 of [1] to our case. For any  $M \in \mathcal{M}$  and any  $(\mathcal{M}, \mu)$ -uniform point set  $\mathcal{P} = \{X_1, \dots, X_N\}$ , we have

$$\frac{1}{N} \sum_{n=1}^N \chi_M(X_n) = \frac{A(M; \mathcal{P})}{N} = \mu(M) = \int_X \chi_M d\mu$$

by the definition of an  $(\mathcal{M}, \mu)$ -uniform point set.

For any  $l \in \mathcal{L}_{\mathcal{M}}$  and any  $(\mathcal{M}, \mu)$ -uniform point set  $\mathcal{P} = \{X_1, \dots, X_N\}$ , we have

$$\int_X l d\mu = \frac{1}{N} \sum_{n=1}^N l(X_n).$$

Thus for any  $f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)$  and any  $(\mathcal{M}, \mu)$ -uniform point set  $\mathcal{P} = \{X_1, \dots, X_N\}$  we have

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^N f(X_n) - \int_X f d\mu \\ &= \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) + \frac{1}{N} \sum_{n=1}^N l(X_n) - \int_X (f - l) d\mu - \int_X l d\mu \\ &= \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) - \int_X (f - l) d\mu \end{aligned}$$

for all  $l \in L_{\mathcal{M}}$ .

So

$$\begin{aligned} & \text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N f(X_n) - \int_X f d\mu \right| \\ & \leq \text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) \right| + \int_X |f - l| d\mu \\ & \leq \text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) \right| + \int_{\{x \in X \mid \|f - l\| > \|f - l\|_\infty\}} |f - l| d\mu \\ & \quad + \int_{\{x \in X \mid \|f - l\| \leq \|f - l\|_\infty\}} |f - l| d\mu \end{aligned}$$

for all  $l \in L_{\mathcal{M}}$ .

Since

$$\begin{aligned} & \{(X_1, \dots, X_N) \in \mathcal{C} \mid \left| \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) \right| > \|f - l\|_\infty\} \\ & \subset \{(X_1, \dots, X_N) \in \underbrace{X \times \dots \times X}_N \mid \left| \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) \right| > \|f - l\|_\infty\} \\ & \subset \cup_{n=1}^N (X \times \dots \times \{X_n \in X \mid |(f - l)(X_n)| > \|f - l\|_\infty\} \times \dots \times X), \end{aligned}$$

we have

$$\begin{aligned} & \underbrace{\mu \times \dots \times \mu}_N (\{(X_1, \dots, X_N) \in \mathcal{C} \mid \left| \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) \right| > \|f - l\|_\infty\}) \\ & \leq \underbrace{\mu \times \dots \times \mu}_N (\{(X_1, \dots, X_N) \in \underbrace{X \times \dots \times X}_N \mid \left| \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) \right| > \|f - l\|_\infty\}) \\ & \leq \sum_{n=1}^N \underbrace{\mu \times \dots \times \mu}_N (X \times \dots \times \{X_n \in X \mid |(f - l)(X_n)| > \|f - l\|_\infty\} \times \dots \times X) \\ & = \sum_{n=1}^N \mu(\{X_n \in X \mid |(f - l)(X_n)| > \|f - l\|_\infty\}) = 0, \end{aligned}$$

the last equality follows from Fubini's theorem (see P.384 of [2]).

From the definition of  $\mathcal{L}_\infty$ -norm, we have

$$\text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N (f - l)(X_n) \right| \leq \|f - l\|_\infty.$$

Also from the definition of  $\mathcal{L}_\infty$ -norm, we have

$$\mu(\{x \in X \mid |f - l| > \|f - l\|_\infty\}) = 0,$$

thus

$$\int_{\{x \in X \mid |f - l| > \|f - l\|_\infty\}} |f - l| d\mu = 0.$$

So

$$\begin{aligned} & \text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N f(X_n) - \int_X f d\mu \right| \\ & \leq \|f - l\|_\infty + \int_{\{x \in X \mid |f - l| \leq \|f - l\|_\infty\}} |f - l| d\mu \\ & \leq 2\|f - l\|_\infty \end{aligned}$$

for all  $l \in L_{\mathcal{M}}$ . □

Let  $\mathcal{M} = \{M_1, \dots, M_k\}$  be a finite nonempty subset of  $\mathcal{A}$  such that  $M_1, \dots, M_k$  are disjoint and  $\cup_{i=1}^k M_i = X$ . If  $f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)$ , then  $f \in \mathcal{L}_\infty(X, \mathcal{A}|_{M_i}, \mu|_{M_i})$  for any  $1 \leq i \leq k$ . Let

$$G_j(f) = \begin{cases} -\inf_{x \in M_j} f^-, & \text{if } \mu(\{x \in M_j \mid f^+(x) > 0\}) = 0; \\ \text{esssup}_{x \in M_j} f^+, & \text{otherwise,} \end{cases}$$

$$g_j(f) = \begin{cases} \inf_{x \in M_j} f^+, & \text{if } \mu(\{x \in M_j \mid f^-(x) > 0\}) = 0; \\ -\text{esssup}_{x \in M_j} f^-, & \text{otherwise,} \end{cases}$$

for  $1 \leq j \leq k$ . Define

$$S_{\mathcal{M}}(f) = \max_{1 \leq j \leq k} (G_j(f) - g_j(f)).$$

**Corollary 2.3.** *Let  $(X, \mathcal{A}, \mu)$  be an arbitrary probability space. Let  $\mathcal{M} = \{M_1, \dots, M_k\}$  be a finite nonempty subset of  $\mathcal{A}$  such that  $M_1, \dots, M_k$  are disjoint and  $\cup_{j=1}^k M_j = X$ . Let*

$$\mathcal{C} = \{(X_1, \dots, X_N) \in \underbrace{X \times \dots \times X}_N \mid \mathcal{P} = \{X_1, \dots, X_N\} \text{ is an } (\mathcal{M}, \mu)\text{-uniform point set}\}.$$

Then for any  $f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)$ , we have

$$\text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N f(X_n) - \int_X f d\mu \right| \leq S_{\mathcal{M}}(f).$$

*Proof.* We extend Niederreiter's proof for Corollary 1 of [1] to our case. Let

$$C_j = \frac{1}{2}(G_j(f) + g_j(f))$$

for  $1 \leq j \leq k$ , let

$$l = \sum_{j=1}^k C_j \chi_{M_j}.$$

Since

$$\{t \in M_j \mid |f(t) - C_j| > \frac{G_j(f) - g_j(f)}{2}\} \subset \{t \in M_j \mid f(t) > G_j(f)\} \cup \{t \in M_j \mid f(t) < g_j(f)\},$$

we have

$$\mu(\{t \in M_j \mid |f(t) - C_j| > \frac{G_j(f) - g_j(f)}{2}\}) \leq \mu(\{t \in M_j \mid f(t) > G_j(f)\}) + \mu(\{t \in M_j \mid f(t) < g_j(f)\}) = 0$$

by the definition of  $\mathcal{L}_\infty$ -norm.

For  $t \in M_j$ , we have

$$\begin{aligned} & \text{esssup}_{t \in M_j} |f(t) - l(t)| \\ &= \text{esssup}_{t \in M_j} |f(t) - C_j| \\ &\leq \frac{1}{2}(G_j(f) - g_j(f)) \end{aligned}$$

Therefore

$$\|f - l\|_\infty \leq \frac{1}{2} S_{\mathcal{M}}(f).$$

Thus

$$D(f, L_{\mathcal{M}}) \leq \frac{1}{2} S_{\mathcal{M}}(f).$$

From Theorem 2.2, we get our result.  $\square$

**Corollary 2.4.** Let  $(X, \mathcal{A}, \mu)$  be an arbitrary probability space, let  $\mathcal{M} = \{M_1, \dots, M_k\}$  be a finite nonempty subset of  $\mathcal{A}$  such that  $M_1, \dots, M_k$  are disjoint and  $\cup_{j=1}^k M_j = X$ . Let

$$\mathcal{C} = \{(X_1, \dots, X_N) \in \underbrace{X \times \dots \times X}_N \mid \mathcal{P} = \{X_1, \dots, X_N\} \text{ is an } (\mathcal{M}, \mu)\text{-uniform point set}\}.$$

Then for any  $f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)$ , we have

$$\text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{n=1}^N f(X_n) - \int_X f d\mu \right| \leq \sum_{j=1}^k \mu(M_j) (G_j(f) - g_j(f)).$$

*Proof.* We extend Niederreiter's proof for Theorem 2 of [1] to our case. From the definition of  $\mathcal{L}_\infty$ -norm, we have

$$\mu(M_j) g_j(f) \leq \int_{M_j} f d\mu \leq \mu(M_j) G_j(f).$$

From the definition of uniform point set, we have

$$\begin{aligned} & \{(X_1, \dots, X_N) \in \mathcal{C} \mid \mu(M_j)g_j(f) > \frac{1}{N} \sum_{\substack{n=1 \\ X_n \in M_j}}^N f(X_n)\} \\ & \subset \cup_{n=1}^N (X \times \dots \times \{X_n \in M_j \mid g_j(f) > f(X_n)\} \dots \times X), \end{aligned}$$

for  $1 \leq j \leq k$ .

Thus

$$\begin{aligned} & \underbrace{\mu \times \dots \times \mu}_N (\{(X_1, \dots, X_N) \in \mathcal{C} \mid \mu(M_j)g_j(f) > \frac{1}{N} \sum_{\substack{n=1 \\ X_n \in M_j}}^N f(X_n)\}) \\ & \leq \sum_{n=1}^N \underbrace{\mu \times \dots \times \mu}_N (X \times \dots \times \{X_n \in M_j \mid g_j(f) > f(X_n)\} \dots \times X) \\ & = \sum_{n=1}^N \mu(\{X_n \in M_j \mid g_j(f) > f(X_n)\}) = 0 \end{aligned}$$

for  $1 \leq j \leq k$ , by the definition of  $g_j(f)$  and Fubini's theorem. Similarly,

$$\underbrace{\mu \times \dots \times \mu}_N (\{(X_1, \dots, X_N) \in \mathcal{C} \mid \mu(M_j)G_j(f) < \frac{1}{N} \sum_{\substack{n=1 \\ X_n \in M_j}}^N f(X_n)\}) = 0.$$

Thus

$$\text{esssup}_{(X_1, \dots, X_N) \in \mathcal{C}} \left| \frac{1}{N} \sum_{\substack{n=1 \\ X_n \in M_j}}^N f(X_n) - \int_{M_j} f d\mu \right| \leq \mu(M_j)(G_j(f) - g_j(f)),$$

for  $1 \leq j \leq k$ .

Finally, from

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^N f(X_n) - \int_X f d\mu \\ & = \sum_{j=1}^k \left( \frac{1}{N} \sum_{\substack{n=1 \\ X_n \in M_j}}^N f(X_n) - \int_{M_j} f d\mu \right), \end{aligned}$$

we get our result.  $\square$

**Acknowledgement:** This work was partially supported by Postdoctoral Science Foundation of China. The authors are grateful to the anonymous referee for his/her helpful comments.

#### REFERENCES

- [1] H.Niederreiter, Error bounds for quasi-Monte Carlo integration with uniform point sets, Journal of computational and applied mathematics 150 (2003), 283-292.
- [2] E.Hewitt and K.Stromberg, Real and abstract analysis, GTM25, Springer-Verlag, Berlin Heidelberg, 1965.

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA  
*E-mail address:* hus04@mails.tsinghua.edu.cn

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA  
*E-mail address:* liyan\_00@mails.tsinghua.edu.cn